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Maximal Domains for Strategy-Proof or Maskin Monotonic Choice Rules*

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Abstract

Domains of individual preferences for which the well-known impossibility theorems of Gibbard-Satterthwaite and Muller-Satterthwaite do not hold are studied. To comprehend the limitations these results imply for social choice rules, we search for the largest domains that are possible. Here, we restrict the domain of individual preferences of precisely one individual. It turns out that, for such domains, the conditions of inseparable pair and of inseparable set yield the only maximal domains on which there exist non-dictatorial, Pareto-efficient and strategy-proof social choice rules. Next, we characterize the maximal domains which allow for Maskin monotone, non-dictatorial and Pareto-efficient social choice rules.

1 Introduction

The two most negative results on the decentralization of social choice functions are, respectively, the Gibbard-Satterthwaite (Gibbard (1973), and Satterthwaite 1975) and the Muller-Satterthwaite (Muller and Satterthwaite,

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1977) theorems. The Gibbard-Satterthwaite theorem states that over an unrestricted domain of linear orderings—and with at least three alternatives—any *surjective* and *strategy-proof* social choice function is *dictatorial*. On the other hand, the Muller-Satterthwaite theorem, by establishing the connections between *strategy-proofness* and *Maskin monotonicity*, states that any *unanimous* and *Maskin monotonic* social choice function is *dictatorial*.

These two theorems have a damaging impact on decentralization. *Strategy-proofness* and *Maskin monotonicity* are, respectively, necessary conditions for dominant strategy implementation and Nash implementation. If a planner restricts himself to institutions corresponding to normal game forms having a unique equilibrium at every preference profile, then only trivial social choice functions are decentralizable.

However, the two results strongly rely on the assumption of unrestricted domain of preferences. Restricted domains have delivered possibility results on strategy-proof and Maskin monotonic social choice functions.¹ For instance, if preferences are quasi-linear with respect to a numeraire good, then Clark-Groves mechanisms are strategy-proof.² If preferences are single-peaked, then generalized median voting rules are strategy-proof (see e.g. Moulin, 1980). Finally, in exchange or production economies, if the domain of preferences is such that there always exist a unique Walrasian equilibrium in the interior of the feasible set, then the Walrasian social choice function is *Maskin monotonic* and obviously *non-dictatorial*.³

In this paper, we work with the abstract social choice model. We are interested in the maximal preference domains under which there exists social choice functions that escape the Gibbard-Satterthwaite and the Muller-Satterthwaite theorems. This question is not new, at least for the *strategy-proof* case. In the literature, the approach that is usually followed is to take a possibility domain and to find the maximal enlargement of this domain so that the possibility result still holds. For instance, this is the case of Barberà, Sonnenschein and Zhou (1994); Barberà, Gul and Stacchetti (1994); Serizawa (1995); Serizawa and Ching (1998); or Masso and Neme (2001). Each paper deals with entire restricted domain of preferences. Moreover, as

¹There is a vast literature providing possibility results in various models. We voluntarily only make reference to a few papers.

²Though, as shown in Green and Laffont (1979), their lack of budget-balancedness cannot be overcome. As a consequence, Clark-Groves mechanisms are not fully efficient.

³Obviously, in such a case, the Walrasian social choice function is not *strategy-proof* (see e.g. Hurwicz, 1972).

far as we are aware of, there are no papers that deals with maximal domains for non-trivial *Maskin monotonic* social choice functions. Closely related but different questions were posed by Kalai and Muller (1978) and Kalai and Ritz (1980). They studied the general conditions for domains which admit the existence of non-dictatorial Arrow-type social welfare functions.

Our approach is different. Instead of allowing that any agent's preference domain be restricted, we restrict the preference domain of precisely one agent. By doing so, we look for necessary and sufficient conditions that guarantee a possibility result. We identify two conditions that are both necessary and sufficient for the existence of non-trivial social choice functions. The social choice function K we identify is a hierarchical rule. In that sense, the rule is almost dictatorial. The conditions we identify are respectively *strategy-proof admissibility* and *Maskin admissibility*.

We then examine the maximality of this domain restrictions. We study, in turn, *strategy-proofness* and *Maskin monotonicity* since the conditions we obtain are different. For *strategy-proofness*, we find that a domain of preferences is a *maximal possibility domain* if and only the restricted set of preferences of agent 1 has one *inseparable pair* or one *inseparable set*. The notion of *inseparable pair* is well-known in relation with non-dictatorial Arrow-type welfare functions (see e.g. Kalai and Ritz, 1980). Just to fix idea, let us briefly discuss the notion of *inseparable pair* and *inseparable set*. We say that an agent has an *inseparable pair* if there exist two alternatives x and y such that whenever x is ranked best, then y is second-best.⁴ Suppose for instance that a board of managers has two vacancies. Current members of the board are contemplating several candidates. Among those are b , a bossy individual, and w a wimp that is afraid of b . Then, an agent i could rank b at the top and w second because he expects w to be obedient and to copy b 's decisions—the power of b would then be increased.⁵ Clearly, agent i has an *inseparable pair* (b, w) .

An agent has an *inseparable set* if there exist a set of alternatives B —of cardinality at least equal to three—such that all preferences with a best alternative in B ranks the alternatives in B adjacent to each other (while

⁴The definition of *inseparable pair* used by Kalai and Ritz (1980) is stronger: it says that for all preferences where x is preferred to y , these two are ranked adjacent to each other.

⁵On the other hand, for different preferences, this agent i may like a strong candidate but dislike b 's bossiness and w 's obedience. Then, b and w are not necessarily ranked adjacent to each other because i does not like when the power of b is increased.

the ranking of alternatives both within B and the complement of B can be reversed from one profile to another).⁶ For instance, suppose that there are two parties, *left* and *right*, involved in an election where several positions have to be distributed (e.g. prime minister versus lower ranked ministers).⁷ Agents have to vote for one of the two parties and have to rank candidates within each party according to the position they would like them to have. There, it is natural to assume that agents rank individuals from the same party adjacent to each other. Then, every agent has an inseparable set.⁸ On the other hand, in order to escape the Gibbard-Satterthwaite theorem, we only need one agent to satisfy the domain restriction imposed by *inseparable pair* or *inseparable set*.

The notion of inseparable set goes at least back to Storcken (1989). It follows that, in case of precisely three alternatives, if one arbitrary chosen preference is no longer admissible for one arbitrary chosen agent—yielding an inseparable pair for this agent—then on this domain neither the negative result of Gibbard Satterthwaite nor that of Muller Satterthwaite can be deduced.

For *Maskin monotonicity*, the condition is more intricate and relies on the existence of disjoint subsets of the sets of alternative and the existence of an asymmetric and transitive relation on the set of alternatives. However, like in the case of the *maximal strategy-proof possibility domains*, the result spells out a characterization of *maximal Maskin monotone possibility domains* in terms of a set of pairs of alternatives on which the coalition of agents whom sets of preferences are not restricted are decisive

The plan of the paper is as follows. In section 2, we introduce the model and the necessary definitions useful for the paper. Section 3 presents sufficient conditions for the existence of possibility results, as well as some connections between *strategy-proofness* and *Maskin monotonicity*. Then, section 4 establishes that these sufficient conditions are also necessary. Section 5 characterizes the *maximal domain* for *strategy-proof*, *efficient* and *non-dictatorial* social choice functions. Next, section 6 characterizes the *maximal domain* for *Maskin monotonic*, *efficient* and *non-dictatorial* social choice functions. Finally, section 7 provides some concluding remarks.

⁶Observe that if the cardinality of B is equal to 2, then we have in fact two inseparable pairs.

⁷Thus, each party has several candidates.

⁸Thus, the *strategy-proof* and *Pareto-efficient* rules that one may identify there will be less hierarchical than the one we identify.

2 The model

There is a set of alternatives $A = \{1, \dots, m\}$, with $m \geq 3$ and a set of agents $N = \{1, \dots, n\}$ with $n \geq 2$.

Agents are endowed with a preference relation R over elements of A that is (strongly) complete, anti-symmetric and transitive; that is a preference relation is a linear order over alternatives. Let $L(A)$ denote the set of all these preferences and $L^N(A)$ be the set of possible preference profiles. For each $R \in L(A)$ and each $(x, y) \in A \times (A \setminus \{x\})$, $x... = R$ means that x is the best alternative at R , $...x...y... = R$ means that x is strictly preferred to y at R , $...xy... = R$ means that x is strictly preferred to y at R and there is no alternative in between; and finally $x...y = R$ means that x is the best alternative and y is the worst alternative at R . Let $L_x(A)$ denote the set of linear orderings R such that $x... = R$.

For each relation R on A , for each $x, y \in A$, $(x, y) \in R$ means that $...x...y... = R$ or $x = y$, i.e. $R \subseteq A \times A$. The upper contour of an alternative x at R is defined as $up(x, R) = \{y \in A : (y, x) \in R\}$ and the lower contour of x at R is defined as $low(x, R) = \{y \in A : (x, y) \in R\}$. The complement of the upper contour of x at R is $non-up(x, R) = A - up(x, R)$. Furthermore, let $I_A = \{(x, x) : x \in A\}$ denote the identity relation on A .

To model restrictions of domains of individual preferences let $\emptyset \neq L^i \subseteq L(A)$ be the domain of individual preferences of agent $i \in N$. From now on, we assume that $L^i = L(A)$ if $i \geq 2$ and $L^1 \subsetneq L(A)$. For each alternative $a \in A$ and set of preferences $V \subseteq L(A)$, let $L_a^i = L_a(A) \cap L^i$ and $V_a = V \cap L_a(A)$. For a *coalition* M , i.e. a subset of N , and alternatives x, y in A , let $L_x^M \times L_y^{N-M}$ denote the set of profiles p such that $x... = p(i)$ for all $i \in M$ and $y... = p(i)$ for all $i \in N - M$.⁹ We introduce the following restriction on domain of preferences.

Convexity: The domain $L^i \subseteq L(A)$ is *convex* if for all R^1, R^2 in L^i and R^3 in $L(A)$,

$$R^1 \cap R^2 \subseteq R^3 \subseteq R^1 \cup R^2 \text{ implies that } R^3 \in L^i.$$

A social choice rule K is a function from L^N to A . The set of profiles L^N is convex if for each agent $i \in N$, the preference domain L^i is convex. For coalitions M and profiles $p, q \in L^N$, the preference profile p is said to be a M -deviation of a profile q if $p|_{N-M} = q|_{N-M}$.

⁹We set $L_x^\emptyset \times L_y^N = L_y^N$ and $L_x^N \times L_y^\emptyset = L_x^N$.

Inseparable pair: The set of preferences L^i has an *inseparable pair* $(x, y) \in A \times (A \setminus \{x\})$ if $xy\dots = R$ for each $R \in L_x^i$.

Inseparable set: Let B be a subset of A , with $3 \leq \#B < m$. The set of preferences L^i has an *inseparable set* B , with $3 \leq \#B < m$, if for each $b \in B$, for each $R \in L_b^i$, for each $a \in B$ and for each $c \in A - B$, we have $\dots a\dots c\dots = R$.

So, for all preferences R in L^i , if the best alternative at R is in B , then B is preferred to $(A - B)$ at R . Note that this condition trivially holds for the empty set, any singleton set and the set A itself. Therefore these are excluded. Furthermore, if B consists of precisely two alternatives, then having an *inseparable set* means having two *inseparable pairs*, which explains why sets with cardinality 2 are excluded in the definition of *inseparable set*.

The notion of *inseparable pair* is well-known see e.g. Kalai and Ritz (1980). Although there it is a slightly stronger condition. That is L^i has an *inseparable pair* (x, y) if for all $R \in L^i$ if $\dots x\dots y\dots = R$, then $\dots xy\dots = R$. Here we only need this inseparability if x is top alternative, because the almost dictatorial rule depends mainly on the top alternatives of agent 1, the agent with the restricted preference set. A similar remark as for inseparable pair holds for inseparable set as defined in Storcken (1989).

The following conditions for choice rules are well-known. We just rephrase these using the notation at hand.

Non-dictatorship: The choice rule K is *non-dictatorial* if for each agent $i \in N$, there is a profile $p \in L^N$ such that $K(p)\dots \neq p(i)$.

Pareto efficiency: The choice rule K is *Pareto-efficient* if for each $(x, y) \in A \times A$ and each $p \in L^N$ such that for all agents $i \in N$, $\dots x\dots y\dots = p(i)$, then $K(p) \neq y$.

Strategy-proofness: The choice rule K is strategy-proof if for each agent $i \in N$ and each $p, q \in L^N$ such that q is an $\{i\}$ -deviation of p , we have that either $K(p) = K(q)$ or $\dots K(p)\dots K(q)\dots = p(i)$.

Intermediate strategy-proofness: The choice rule K is *intermediate strategy-proof* if for each coalition $M \subseteq N$ and for each profile $p \in L^N$, such that there is a preference $R \in L(A)$ with $p(i) = R$ for all $i \in M$, and all M -deviations $q \in L^N$, it holds that $\dots K(p)\dots K(q)\dots = R$ or $K(p) = K(q)$.

Maskin monotonicity: The choice rule K is *Maskin monotonic* if for each $p, q \in L^N$,

$low(K(p), p(i)) \subseteq low(K(p), q(i))$ for each $i \in N$ implies that $K(p) = K(q)$.

Strong positive association: The choice rule K is *strongly positively associated* if for each $p, q \in L(A)$ and each $a \in A$,

$low(a, p(i)) \subseteq low(a, q(i))$ for each $i \in N$ and $p(i)|_{A-\{a\}} = q(i)|_{A-\{a\}}$ implies that $K(q) \in \{K(p), a\}$.

Decisiveness of coalitions appear also in our setting as a powerful tool to analyze the problem at hand.

Decisiveness: At choice rule K , a coalition $M \subseteq N$ is said to be *decisive* on $(x, y) \in A \times A$, if $K(p) = x$ for each profile $p \in L_x^M \times L_y^{N-M}$.

Let $D_K(M) = \{(x, y) \in A \times A : M \text{ is decisive on } (x, y)\}$. If K is *Pareto-efficient*, then it follows immediately that $I_A \subseteq D_K(M)$. It appears that the rules which simultaneously satisfy *Pareto-efficiency*, *non-dictatorship* and either *strategy-proofness* or *Maskin monotonicity* on the domains at hand are almost dictatorial.

Let $I_A \subsetneq D \subsetneq A \times A$ for some transitive relation D on A . Define by K_D the hierarchical choice rule corresponding to D defined as follows: for every profile p such that $p(1) \in L_y^1$, $K_D(p) = best(p(2)|_{up(y, D)})$. Because of D being unequal to both I_A and $A \times A$, it follows that K_D is *non-dictatorial*. Transitivity of D imposes a kind of rationality which under *Pareto-efficiency*¹⁰ is implied by both *Maskin-monotonicity* and *strategy-proofness*. Next, we discuss conditions that D has to satisfy in order for K_D to be a *strategy-proof* or *Maskin monotonic* social choice rule. We introduce the following definitions.

Strategy-proof admissibility: The pair (D, L^1) is *strategy-proof admissible* if $I_A \subsetneq D \subsetneq A \times A$, D is transitive and for each $(x, y) \in D$, each $R \in L_y^1$ and each $z \in up(x, R)$, both

$$\begin{aligned} (z, y) &\in D \text{ and} \\ (x, z) &\in D \text{ whenever } L_z^1 \neq \emptyset. \end{aligned}$$

¹⁰But independent of the actual preference domain restriction of agent 1

Let $strategy(D, L^1)$ stands for the *strategy-proof admissibility* of the pair (D, L^1) .

Maskin admissibility: The pair (D, L^1) is *Maskin admissible*, notation $Maskin(D, L^1)$ if $I_A \subsetneq D \subsetneq A \times A$, D is transitive and for all $(x, y) \in D$ all $R \in L_y^1$ all $z \in up(x, R)$ and all $R' \in L_z^1$ both

$$\begin{aligned} (z, y) &\in D \text{ and} \\ (x, z) &\in D \text{ whenever } low(x, R) \subseteq low(x, R'). \end{aligned}$$

Let $Maskin(D, L^1)$ stands for the *Maskin admissibility* of the pair (D, L^1) .

3 Sufficient conditions, strategy-proofness and Maskin monotonicity

The following Lemma shows that the condition of *strategy-proof admissibility* and *Maskin admissibility* are sufficient to guarantee that K_D is *strategy-proof* or *Maskin-monotone* respectively and therewith explains the names of these two requirements on D and L^1 .

Lemma 1 *Consider the hierarchical choice rule $K_D : L^N \rightarrow A$, corresponding to D . Then*

1. *$strategy(D, L^1)$ implies that K_D is non-dictatorial, Pareto-efficient and strategy-proof;*
2. *$Maskin(D, L^1)$ implies that K_D is non-dictatorial, Pareto-efficient and Maskin-monotone.*

Proof. *Non-dictatorship* follows in both cases because of $I_A \subsetneq D \subsetneq A \times A$. To prove *Pareto-efficiency* let p be a profile with $p(1) \in L_y^1$. It is sufficient to prove that *Pareto-efficiency* is satisfied at this profile p . This is indeed the case whenever $K_D(p) = y$. So, suppose that $K_D(p) = best(p(2)|_{up(y, D)})$ and $K_D(p) \neq y$. Because of $strategy(D, L^1)$ as well as $Maskin(D, L^1)$, it follows that $z \in up(y, D)$ if there are $x \in up(y, D)$ and $R \in L_y^1$ with $y...z...x... = R$. So, for all $a \in non-up(y, D)$, we have $...K_D(p)...a... = p(1)$ and for all $a \in up(y, D) - \{K_D(p)\}$, we have $...K_D(p)...a... = p(2)$. So, *Pareto-efficiency* is satisfied at p .

In order to prove (1), assume $strategy(D, L^1)$. Furthermore, let p be as above and profile q an $\{i\}$ -deviation of p for some agent i with $q(1) \in L_z^1$ for some $z \in A$. To the contrary suppose that $\dots K_D(q) \dots K_D(p) \dots = p(i)$. By the definition of K_D it follows that $i \leq 2$. If $i = 2$, then $p(1) = q(1)$. Consequently, $y = z$ and $K_D(q) = best(q(2)|_{up(z,D)}) = best(q(2)|_{up(y,D)})$. So, in that case $K_D(q) \in up(y, D)$. Because $K_D(p) = best(p(2)|_{up(y,D)})$, we obtain a contradiction with $\dots K_D(q) \dots K_D(p) \dots = p(i)$. To conclude the proof of (1), let $i = 1$. Therefore, $p(2) = q(2)$. Because of $\dots K_D(q) \dots K_D(p) \dots = p(1)$ and $strategy(D, L^1)$, it follows that $K_D(p) \in up(K_D(q), D)$. Thus, since D is transitive, we conclude that $K_D(p) \in up(z, D)$. But then $K_D(q) = best(q(2)|_{up(z,D)}) = best(p(2)|_{up(z,D)})$ and $\dots K_D(q) \dots K_D(p) \dots = p(1)$ contradict each other. This proves *strategy-proofness*.

In order to prove (2), assume $Maskin(D, L^1)$ and consider $\{i\}$ -deviations p and q as before with, in addition, $low(K_D(p), p(i)) \subseteq low(K_D(p), q(i))$. It is sufficient to prove that $K_D(p) = K_D(q)$. This obviously holds whenever $i \geq 2$. So suppose that $i = 1$. Since $low(K_D(p), p(1)) \subseteq low(K_D(p), q(1))$, it follows that $z \in up(K_D(p), p(1))$. $Maskin(D, L^1)$, or reflexivity of D , or $z = y$ imply that $(K_D(p), z) \in D$ and $(z, y) \in D$. Furthermore, transitivity of D implies that $up(z, D) \subseteq up(y, D)$. But because $K_D(p) = best(p(2)|_{up(y,D)})$, it follows that $K_D(p) = best(p(2)|_{up(y,D)}) = best(p(2)|_{up(z,D)}) = best(q(2)|_{up(z,D)}) = K_D(q)$. ■

A domain L^N is called a *strategy-proof possibility domain* if there exist choice rules $K : L^N \rightarrow A$ which are simultaneously *non-dictatorial*, *strategy-proof* and *Pareto-efficient*. Furthermore, it is called a *maximal strategy-proof possibility domain* if it is a *strategy-proof possibility domain* and there is no other *strategy-proof possibility domain*, say \hat{L}^N , such that $L^N \subsetneq \hat{L}^N$. Similarly we define a *Maskin-monotone possibility domain* and a *maximal Maskin-monotone possibility domain* by replacing the condition of *strategy-proofness* by the condition of *Maskin monotonicity* in the two previous definitions.

Example 1 In case L^1 has an inseparable pair (y, x) or an inseparable set B , it follows straightforwardly that $strategy(D, L^1)$, where $D = \{(x, y)\}$ or $D = B \times B$ respectively. So, the previous Lemma implies that in these situations, the hierarchical choice rule K_D is non-dictatorial, Pareto-efficient and strategy-proof. Thus, in that case, L^N is a strategy-proof possibility domain.

The following results logically link several of the conditions defined above.

Theorem 1 Let $K : L^N \rightarrow A$ be a social choice rule. Then

1. K is strategy-proof if and only if it is intermediate strategy-proof;
2. If K is strategy-proof, then it is Maskin monotonic and strongly positively associated;
3. If L^N is convex, then strategy-proofness of K is equivalent to Maskin monotonicity of K , as well as to strong positive association of K .

Proof. (First part) Clearly *intermediate strategy-proofness* implies *strategy-proofness*. So, suppose K is *strategy-proof*. Furthermore, let p and q be S -deviations for some $S \subseteq N$ such that $p(i) = R$ for some $R \in L(A)$. It is sufficient to prove that $\dots K(q) \dots K(p) \dots \neq R$. Without loss of generality let $S = \{1, 2, \dots, s\}$. Take profiles $r^0, r^1, r^2, \dots, r^s \in L^N$ defined for all $i \in N$ by $r^t(i) = p(i)$ if $i > t$, and by $r^t(i) = q(i)$ otherwise. So, $r^0 = p$ and $r^s = q$. By *strategy-proofness*, it follows that $\dots K(r^{t+1}) \dots K(r^t) \dots \neq r^t(t+1) = R$ for all $t \in \{0, 1, 2, \dots, s-1\}$. Because of negative transitivity of R , it follows that $\dots K(q) \dots K(p) \dots \neq R$.

(Second part) Let p and q be $\{i\}$ -deviations in L^N such that,

$$low(K(p), p(i)) \subseteq low(K(p), q(i)).$$

It is sufficient to prove that $K(p) = K(q)$. By *strategy-proofness*, it follows that $K(q) \in low(K(p), p(i))$ and $K(p) \in low(K(q), q(i))$. Given our assumption, we have that $K(q) \in low(K(p), q(i))$ and $K(p) \in low(K(q), q(i))$. As $q(i)$ is antisymmetric we have that $K(p) = K(q)$.

In order to prove *strong positive association*, let p and q be $\{i\}$ -deviations in L^N . Furthermore, let $a \in A$ be such that $low(a, p(i)) \subseteq low(a, q(i))$ and $p(i)|_{A-\{a\}} = q(i)|_{A-\{a\}}$. It is sufficient to prove that $K(q) \in \{a, K(p)\}$. Suppose that $K(p) \neq K(q)$. Then we have to prove that $K(q) = a$. By *strategy-proofness*, it follows that $\dots K(p) \dots K(q) \dots = p(i)$ and $\dots K(q) \dots K(p) \dots = q(i)$. Because $p(i)|_{A-\{a\}} = q(i)|_{A-\{a\}}$ and $low(a, p(i)) \subseteq low(a, q(i))$, this can only hold if $K(q) = a$.

(Third part) Let p and q be two $\{i\}$ -deviations in L^N . It is sufficient that both *Maskin monotonicity* and *strong positive association* imply that,

$$K(q) \in low(K(p), p(i)).$$

Because of convexity of L^N there are $R^0, R^1, R^2, \dots, R^k \in L^i$ such that $R^0 = p(i)$, $R^k = q(i)$, $R^t \Delta R^{t+1} = \{(x_t, y_t), (y_t, x_t)\}$ for some x_t and y_t in A , with

$x_t \neq y_t$, for all $t \in \{0, 1, 2, \dots, k-1\}$ and $\dots y_t \dots x_t \dots = R^s$ for all $t < s \leq k$ and $\dots x_t \dots y_t \dots = R^s$ for all $0 \leq s \leq t$. Now take profiles $r^0, r^1, r^2, \dots, r^k \in L^N$, $\{i\}$ -deviation of p such that $r^t(i) = R^t$ for all $t \in \{0, 1, 2, \dots, k\}$. Now if $K(r^t) \neq K(r^{t+1})$ for some $t \in \{0, 1, 2, \dots, k-1\}$, then *Maskin monotonicity* as well as *strong positive association* imply that $K(r^t) = x_t$ and $K(r^{t+1}) = y_t$. Hence $K(r^{t+1}) \in \text{low}(K(r^t), p(i))$. By transitivity, it follows that $K(r^k) \in \text{low}(K(r^0), p(i))$. So, $K(q) \in \text{low}(K(p), p(i))$ proving that K is *strategy-proof*. ■

4 Necessary conditions for strategy-proofness or Maskin-monotonicity

In the previous section, the conditions of *strategy-proof admissibility* and of *Maskin admissibility* appeared to be sufficient to guarantee the existence of choice rules on the restricted domain that are *Pareto-efficient*, *non-dictatorial* and *strategy-proof*, or *Maskin monotonic*, respectively. Here we prove that these two conditions are in fact also necessary.

For the rest of this section, let K be a *Pareto-efficient* and *non-dictatorial* social choice rule. We will show that whenever K is *strategy-proof* or *Maskin monotonic*, then $I_A \subsetneq D_K(N - \{1\}) \subsetneq A \times A$ is transitive and that the pair $(D_K(N - \{1\}), L^1)$ is respectively *strategy-proof admissible* or *Maskin admissible*.

To avoid needless repetitions, assume that K is at least *Maskin monotonic*. So, cases at which the social choice rule K is *strategy-proof* are spelled out explicitly in this section.

Lemma 2 *Let $p \in L^N$, $M \subseteq N$ and $(x, y) \in A \times (A \setminus \{x\})$ such that for each agent $i \in M$, $xy\dots = p(i)$ and for each agent $i \in N - M$, $y\dots = p(i)$. Let $K(p) = x$. Then $(x, y) \in D_K(M)$.*

Proof. Let $q \in L_x^M \times L_y^{N-M}$. It is sufficient to prove that $K(q) = x$. Let $r \in L^N$ be an $(N - M)$ -deviation of p and an M -deviation of q . Now, *Pareto-efficiency* implies that $K(r) \in \{x, y\}$. If $K(r) = y$, then *Maskin monotonicity* would imply the contradiction that $K(p) = y$. Therefore, $K(r) = x$. But then *Maskin monotonicity* implies that $K(q) = x$. ■

Lemma 3 *$I_A \subsetneq D_K(N - \{1\}) \subsetneq A \times A$ and $D_K(N - \{1\})$ is transitive.*

Proof. (*Proof of $I_A \subsetneq D_K(N - \{1\})$*) To the contrary, suppose that $I_A = D_K(N - \{1\})$. We show that this leads to the contradiction that agent 1 is a dictator. Consider $R \in L^1$ and a profile $p \in L^N$, with $xy\dots = R = p(1)$ for some x and y . To deduce the contradiction, it is sufficient to prove that $K(p) = x$. Consider the profiles q and r —both $N - \{1\}$ -deviations of p —such that $q(i) = yx\dots$ and $r(i) = y\dots x$ for each $i \geq 2$. Because of *Pareto-efficiency*, it follows that $K(q) \in \{x, y\}$. Now, by Lemma 2, it follows that $K(q) \neq y$, otherwise $(y, x) \in (D_K(N - \{1\}) - I_A)$. But then $K(q) = x$ and $(x, y) \in D_K(\{1\})$. So, $K(r) = x$. By *Maskin monotonicity*, we obtain that $K(p) = x$.

(*Proof of $D_K(N - \{1\}) \subsetneq A \times A$*) To the contrary suppose that $D_K(N - \{1\}) = A \times A$. We show that this leads to the contradiction that K is *dictatorial*. For $R \in L^1$, consider the social choice rule K^R defined by $K^R(p) = K(R, p)$ for each profile $p \in L(A)^{N-\{1\}}$. It is clear that K^R is surjective—and even unanimous—and *Maskin monotonic*. By Theorem 1, it is *strategy-proof*. Hence, by the Gibbard and Satterthwaite Theorem, it follows that $K^R(p)$ is *dictatorial*, say by agent $i_R \geq 2$.

Consider two preferences R and R' in L^1 . In order to prove that K is *dictatorial*, it is sufficient to show that $i_R = i_{R'}$. To the contrary, suppose that $i_R \neq i_{R'}$. We deduce a contradiction and are done. Obviously there are different alternatives $x, y, z_1, z_2, \dots, z_{k-1}$ and z_k , where k may be zero, such that $z_1 z_2 \dots z_k x \dots = R$ and $z_1 z_2 \dots z_k y \dots = R'$. So x and y are the first alternatives on which the preferences R and R' differ. Consider profiles p and q which are $\{1\}$ -deviations such that $p(1) = R$, $q(1) = R'$, $p(i_R) = q(i_R) = y\dots$ and $p(i_{R'}) = q(i_{R'}) = x\dots$. Then since i_R is a dictator at K^R , it follows that $K(p) = y$ and because $i_{R'}$ is a dictator at $K^{R'}$, it follows that $K(q) = x$. Finally, because $\text{low}(y, R) \subseteq \text{low}(y, R')$, we have a contradiction with *Maskin monotonicity* of K .

(*Proof of transitivity*) Let $(x, y), (y, z) \in D_K(N - \{1\})$. It is sufficient to prove that $(x, z) \in D_K(N - \{1\})$. This is trivially the case when x, y and z are not three different alternatives. So, let these three alternatives be different. Let $R = z\dots \in L^1$ arbitrary. Consider $p \in L^N$ such that $p(1) = R$ and $p(i) = xz\dots$ for each $i \in N - \{1\}$. By Lemma 2, it is sufficient to prove that $K(p) = x$. Consider $(N - \{1\})$ -deviation q of p such that $xyz\dots = q(i)$ for all $i \in N - \{1\}$. If $K(q) = y$, then *Maskin monotonicity* would yield $K(r) = y$ —where r is the $\{1\}$ -deviation of q with $r(1) = y\dots$ —which contradicts $(x, y) \in D(N - \{1\})$. If $K(q) = z$, then *Maskin monotonicity* would yield $K(u) = z$, where u is a $(N - \{1\})$ -deviation of q , with $yz\dots = u(i)$ for all

$i \in N - \{1\}$, which contradicts $(y, z) \in D(N - \{1\})$. Therefore $K(q) = x$ and *Maskin monotonicity* implies that $K(p) = x$. ■

Lemma 4 *Let x, y and z be different alternatives and $R \in L^1$ such that $y...z.....x... = R$. Let $(x, y) \in D_K(N - \{1\})$. Then*

1. $(z, y) \in D_K(N - \{1\})$;
2. $(x, z) \in D_K(N - \{1\})$ if K is strategy-proof and $L_z^1 \neq \emptyset$;
3. $(x, z) \in D_K(N - \{1\})$ if there are $R' \in L_z^1$ such that $\text{low}(x, R) \subseteq \text{low}(x, R')$.

Proof. (*Proof of 1*) Let $q \in L^N$ such that $q(1) = R$ and $zy... = q(i)$ for all $i \in N - \{1\}$. By Lemma 2 it is sufficient to prove that $K(q) = z$. Consider p and r , two $(N - \{1\})$ -deviations of q , such that $xzy... = p(i)$ and $zxy... = r(i)$ for all $i \in N - \{1\}$. Because of $(x, y) \in D(N - \{1\})$, it follows that $K(p) = x$. *Pareto-efficiency* implies that $K(r) \in \{y, z\}$. If $K(r) = y$, then *Maskin monotonicity* would yield the contradiction that $K(p) = y$. Therefore, $K(r) = z$ and *Maskin monotonicity* implies that $K(q) = z$.

(*Proof of 2*) Suppose K is *strategy-proof* and $z... = R' \in L_z^1 \neq \emptyset$. Let $q \in L^N$ such that $q(1) = R'$ and $xz... = q(i)$ for all $i \in N - \{1\}$. By Lemma 2, it is sufficient to prove that $K(q) = x$. Consider $(N - \{1\})$ -deviations p of q , with $p(1) = R$. Because of $(x, y) \in D(N - \{1\})$, it follows that $K(p) = x$. *Pareto-efficiency* implies that $K(q) \in \{x, z\}$. Now considering p and q , it follows that $K(q) = z$ would violate *strategy-proofness*. In consequence, $K(q) = x$.

(*Proof of 3*) Suppose $R' \in L_z^1$ such that $\text{low}(x, R) \subseteq \text{low}(x, R')$. Let $q \in L^N$ be such that $q(1) = R'$ and $xz... = q(i)$ for all $i \in N - \{1\}$. By Lemma 2, it is sufficient to prove that $K(q) = x$. Consider $(N - \{1\})$ -deviations p of q , with $p(1) = R$. Because of $(x, y) \in D(N - \{1\})$, it follows that $K(p) = x$. Next, because $\text{low}(x, R) \subseteq \text{low}(x, R')$ and *Maskin monotonicity*, we have that $K(q) = x$. ■

Combining Lemma's 1, 2, 3 and 4 yields the following corollaries.

Corollary 1 *If L^N is a Maskin monotonic possibility domain, then $(D_K(N - \{1\}), L^1)$ is Maskin admissible;*

Corollary 2 *If L^N is a strategy-proof possibility domain, then $(D_K(N - \{1\}), L^1)$ is strategy-proof admissible.*

5 Maximal domains for strategy-proofness

Corollary 2 characterizes *strategy-proof possibility domains* in terms of a set of pairs of alternatives on which the coalition of agents whose sets of preferences are not restricted are decisive. Consider a *Pareto-efficient, strategy-proof* and *non-dictatorial* social choice rule K , such that for all *strategy-proof possibility domains* \widehat{L}^N with $L^N \subseteq \widehat{L}^N$, we have that $L^N = \widehat{L}^N$. So, L^N is a *maximal strategy-proof possibility domain*. We shall prove in Theorem 2 that the conditions on $D_K(N - \{1\})$ under this maximality property yield that L^1 either has one *inseparable pair* or one *inseparable set*. Furthermore, we deduce that these separabilities are not only necessary but also sufficient. The following Lemmas are needed for the proof of Theorem 2.

Lemma 5 *Let $x \in A$. Then $L_x^1 \neq \emptyset$.*

Proof. To the contrary let $L_x^1 = \emptyset$. Take $\widehat{L}^1 = L^1 \cup \{R \in L(A) : xy\dots = R\}$ for some fixed $y \in A - \{x\}$. Clearly, by taking $\widehat{L}^i = L(A)$ for $i > 1$, we obtain that $L^N \subsetneq \widehat{L}^N$. By lemma 1, the latter is a *strategy-proof possibility domain*. This yields a contradiction with L^N being maximal. ■

Intuitively it is reasonable that the smaller the set of decisive pairs of $N - \{1\}$, the larger L^1 can be taken. The following Lemma shows that we may shrink the set of decisive pairs of $N - \{1\}$.

Lemma 6 *Let $(x^*, y^*) \in D_K(N - \{1\})$, with $x^* \neq y^*$.*

Take,

$$Y = \{a \in A : a = y^* \text{ or } (a, y^*) \in D_K(N - \{1\})\} \text{ and,}$$

$$Z = \{a \in A : (a, y^*) \in D_K(N - \{1\}) \text{ and } (y^*, a) \notin D_K(N - \{1\})\}.$$

$$\text{Let } D^* = [(Z \times Z) \cup (Z \times Y) \cup (Y \times Y) \cup I_A] \cap D_K(N - \{1\}).$$

Then strategy(D^, L^1). Furthermore, $[(Z \times Y) \cup (Y \times Y) \cup I_A] \subseteq D_K(N - \{1\})$.*

Proof. By definition $(x^*, y^*) \in D^* - I_A$ and $D^* \subseteq D_K(N - \{1\})$. So, $I_A \subsetneq D^* \subsetneq A \times A$. Because of transitivity of $D_K(N - \{1\})$ and $[(Z \times Z) \cup (Z \times Y) \cup (Y \times Y) \cup I_A]$ it follows that D^* is transitive. Let $x, y, z \in A$, with $\#\{x, y, z\} = 3$. Let $(x, y) \in D^*$. Let $y\dots z\dots x\dots = R$ in L^1 . It is sufficient to

prove that $(z, y) \in D^*$ and $(x, z) \in D^*$. Because of $strategy(D_K(N - \{1\}), L^1)$ and Lemma 5, it follows that $(x, z), (z, y) \in D_K(N - \{1\})$. Moreover, by transitivity of $D_K(N - \{1\})$ and the definition of Y and Z , it follows that $D_K(N - \{1\}) \cap (Y \times Z) = \emptyset$. Thus, it is sufficient to prove that $z \in Y \cup Z$. Because of $y \in Y \cup Z$, it follows that $(y, y^*) \in D_K(N - \{1\})$. However, $(z, y) \in D_K(N - \{1\})$ and transitivity of $D_K(N - \{1\})$ together imply that $(z, y^*) \in D_K(N - \{1\})$. Therefore, $z \in Y \cup Z$.

The furthermore part follows by transitivity of $D_K(N - \{1\})$ and the definition of Y and Z . ■

Lemma 7 *There are disjoint subsets Y and Z of A such that*

1. Y is non-empty, $Y \cup Z \neq A$, and $\#(Y \cup Z) \geq 2$;
2. $I_A \subsetneq [((Y \cup Z) \times Y) \cup I_A] \subseteq D_K(N - \{1\}) \subsetneq A \times A$ and
3. $strategy(((Y \cup Z) \times Y) \cup I_A, L^1)$.

Proof. In Lemma 6, we proved that there are disjoint Y and Z such that $strategy([(Z \times Z) \cup (Z \times Y) \cup (Y \times Y) \cup I_A] \cap D_K(N - \{1\}), L^1)$. Because of the transitivity of $D_K(N - \{1\})$, we may take $(x^*, y^*) \in D_K(N - \{1\})$ such that $x^* \neq y^*$ and $(Z \times Z) \cap D_K(N - \{1\}) \subseteq I_A$. By the furthermore part of the previous Lemma 6, we obtain that $(Y \cup Z) \times Y \subseteq D_K(N - \{1\})$. Hence, it follows that $strategy(((Y \cup Z) \times Y) \cup I_A, L^1)$. The inclusions at (2) follow readily.

Also, by definition, Y is non-empty and because of $x^*, y^* \in Y \cup Z$, with $x^* \neq y^*$, we have $\#(Y \cup Z) \geq 2$.

It remains to prove that $Y \cup Z \neq A$. In order to do so, suppose that $Y \cup Z = A$. We prove that we may take $Z = \emptyset$ and that $Y \neq A$. Consider $y \in Y$ and $R = y...a$ in L^1 where $a \in A$. Because of $strategy(((Y \cup Z) \times Y) \cup I_A, L^1)$ and $(a, y) \in ((Y \cup Z) \times Y)$, it follows for each $b \in A - \{y, a\}$ that $(a, b) \in ((Y \cup Z) \times Y)$. Hence, $b \in Y$ for each $b \in A - \{a, y\}$. Next, because $[((Y \cup Z) \times Y) \cup I_A] \subsetneq A \times A$, it follows that $Z \neq \emptyset$. But then $Z = \{a\}$. As the previous holds for every $a' \in A$ for which there are $y \in Y$, $R \in L_y^1$ with $y...a' = R$ and Z is a singleton, it follows that for each $y \in Y$ and each $R \in L_y^1$ that $y...a = R$. Hence, $strategy((Y \times Y) \cup I_A, L^1)$ and as Z is a singleton $Y \neq A$. By taking $Z = \emptyset$ this shows the existence of such Y and Z . ■

Theorem 2 Consider L^N with $L^i = L(A)$ for $i \geq 2$. Then L^N is a maximal strategy-proof possibility domain if and only if one of the following two holds:

1. There are $a, b \in A$ with $a \neq b$ and $L^1 = V$ where $V = \{R \in L(A) : \text{if } R \in L_a^1, \text{ then } ab\dots = R\}$ which means that L^1 has an inseparable pair (a, b) ;
2. There is a non-trivial subset Y of A , with $\#Y \geq 3$, such that $L^1 = W$ where $W = \{R \in L(A) : \text{if } R \in L_y^1 \text{ for some } y \in Y, \text{ then for all } a \in Y \text{ and all } b \in A - Y \text{ we have } \dots a \dots b \dots = R\}$ which means that L^1 has an inseparable set Y .

Proof. (Only-if-part) Suppose L^1 is a maximal strategy-proof possibility domain. By the previous Lemma there are disjoint subsets Y and Z of A such that

1. Y is non-empty, $Y \cup Z \neq A$, and $\#(Y \cup Z) \geq 2$;
2. $I_A \subsetneq [((Y \cup Z) \times Y) \cup I_A] \subsetneq A \times A$ and
3. $\text{strategy}(((Y \cup Z) \times Y) \cup I_A, L^1)$.

Consider $y \in Y$ and $z, t \in Y \cup Z$. Take $R \in L_y^1$ which exist by Lemma 5 and assume without loss of generality that $y\dots t\dots z\dots = R$. Then by $\text{strategy}(((Y \cup Z) \times Y) \cup I_A, L^1)$, it follows that $(z, t) \in (Y \cup Z) \times Y$. Hence, $t \in Y$. So, $\#Z \leq 1$. Moreover, if $z \in Z$, then for all $t \in Y - \{y\}$, we have $y\dots t\dots z\dots = R$. Because y is chosen arbitrarily, this means that $\text{strategy}((Y \times Y) \cup I_A, L^1)$ or $\#Y = \#Z = 1$. Suppose $Y = \{b\}$ and $Z = \{a\}$. Then $\text{strategy}(((Y \cup Z) \times Y) \cup I_A, L^1)$ implies that L^1 has an inseparable pair (a, b) . Consider the set V . We shall prove $L^1 = V$. Because obviously $\text{strategy}(((Y \cup Z) \times Y) \cup I_A, V)$ and therefore $\text{strategy}(((Y \cup Z) \times Y) \cup I_A, L^1 \cup V)$ it follows by the maximality of L^1 that $V \subseteq L^1$. Now V is defined such that it contains all sets of preferences which have an inseparable pair (a, b) . Therefore $L^1 \subseteq V$. So, $L^1 = V$. Now suppose $\text{strategy}((Y \times Y) \cup I_A, L^1)$. Then it follows that L^1 has an inseparable set Y . Consider the set W . By proving that $L^1 = W$, we end the proof of the only-if-part. Because obviously $\text{strategy}((Y \times Y) \cup I_A, W)$ and therefore obviously $\text{strategy}((Y \times Y) \cup I_A, L^1 \cup W)$, it follows by the maximality of L^1 that $W \subseteq L^1$. But W is defined such that it contains all sets of preferences which have an inseparable set Y . Therefore $L^1 \subseteq W$. So, $L^1 = W$.

(If-part) By example 1 it is clear that if L^1 equals either V or W , then L^N is a *strategy-proof possibility domain*. It remains to prove the maximality of it. Suppose $L^i \subseteq \widehat{L}^i$ for all agents i and \widehat{L}^N is a *maximal strategy-proof possibility domain*. It is sufficient to prove that $L^N = \widehat{L}^N$. By the only-if-part, it follows that \widehat{L}^1 has either an *inseparable set* say \widehat{Y} or an *inseparable pair* say $(\widehat{a}, \widehat{b})$. Because $L^1 \subseteq \widehat{L}^1$ and is such that it contains all sets of preferences which either have an *inseparable set* Y or an *inseparable pair* (a, b) , it follows that the inseparable sets or pairs are equal and that $L^1 = \widehat{L}^1$. ■

6 Maximal domains for Maskin-monotonicity

In this section we characterize the *maximal Maskin monotone possibility domains* for the case that precisely one agent's set of preferences is restricted. Theorem 3 spells out a characterization of *Maskin monotone possibility domains* in terms of a set of pairs of alternatives on which the coalition of agents whom sets of preferences are not restricted are decisive.

Remark 1 Consider the Lemma's 5', 6' and 7' obtained from Lemma's 5, 6 and 7 by replacing the word "strategy" by the word "Maskin" respectively. The proofs of these Lemma's follow likewise by the same substitution in the proofs of the original Lemma's. To avoid obvious repetitions neither the Lemma's 5', 6' and 7' nor their proofs are written out here.

Theorem 3 Let $N = \{1, \dots, n\}$ and $n \geq 2$. Let $L^i = L(A)$ for all $i \geq 2$. Then L^N is a maximal Maskin monotone possibility domain if and only if there are disjoint subsets Y and Z of A and an asymmetric and transitive relation P on A such that

- a Y is non-empty, $Y \cup Z \neq A$, $\#(Y \cup Z) \geq 2$ and $[(Y \cup Z) \times (A - (Y \cup Z))] \subseteq P \subseteq [(Y \cup Z) \times (A - Y)]$;
- b if L^1 has a separable pair (b, a) then $Y = \{b\}$ and $Z = \{a\}$;
- c there is no partition X_1, X_2 of $Y \cup Z$ with $\#X_1 \geq 2$ and $X_1 \times X_2 \subseteq P$ and
- d $L^1 = V$ where $V = \{R \in L(A) :$

1. if $y \dots = R$ for some $y \in Y$, then $P \subseteq R$;

2. if $z... = R$ for some $z \in Z$, then there are $t \in A - (Y \cup Z)$ such that $[(non-up(z, P) - \{z\}) \cap (Y \cup Z)] \subseteq low(t, R)$.

Proof. (*Only-if-part*) Let L^N be a *maximal Maskin monotone possibility domain*. We will show the existence of sets Y, Z and relation P satisfying the conditions formulated in the theorem. By Lemma 7' we have $Y, Z \subseteq A$, where $Y \cap Z = \emptyset, Y \neq \emptyset, Y \cup Z \neq A$ and $Maskin(((Y \cup Z) \times Y) \cup I_A, L^1)$. For different alternatives a and b , define $(a, b) \in P$ if for all $R \in L_y^1$ and all $y \in Y$ we have that $...a...b... = R$. By definition, P is asymmetric and because the preferences in L^1 are transitive, P is transitive. Because of $Maskin(((Y \cup Z) \times Y) \cup I_A, L^1)$, it follows that $[(Y \cup Z) \times (A - (Y \cup Z))] \subseteq P$. Because of the definition of P and Lemma 5' it follows that $P \subseteq [(Y \cup Z) \times (A - Y)]$. This proves part (a).

Claim 1 $Maskin(((Y \cup Z) \times Y) \cup I_A, V)$.

We have to prove the following implications for $x, y, z \in A$ and $R, R' \in V$.

1. If $(x, y) \in (Y \cup Z) \times Y$ and $y...z...x... = R$, then $(z, y) \in (Y \cup Z) \times Y$;
2. If $(x, y) \in (Y \cup Z) \times Y$, $y...z...x... = R$, $z... = R'$ and $low(x, R) \subseteq low(x, R')$, then $(x, z) \in (Y \cup Z) \times Y$.

The proof of the first implication follows immediately from the first condition in the definition of V and $[(Y \cup Z) \times (A - (Y \cup Z))] \subseteq P$. To prove the second implication, let x, y, z, R and R' be as in the premises of implication 2. It is sufficient to prove that $low(x, R) \not\subseteq low(x, R')$ or $(x, z) \in (Y \cup Z) \times Y$. Clearly by implication 1, it follows that $z \in (Y \cup Z)$. If $z \in Y$, then evidently $(x, z) \in (Y \cup Z) \times Y$. So, suppose $z \notin Y$, which implies that $z \in Z$. Now because of $P \subseteq R$ and the definition of P , it follows that $(x, z) \notin P$. Of course $x \neq z$. So, there are $t \in A - (Y \cup Z)$ such that $x \in low(t, R')$. Now, since $t \in A - (Y \cup Z)$, it follows that $y...x...t... = R$. Thus, $low(x, R) \not\subseteq low(x, R')$. This proves the second implication and ends the proof of the claim.

Next we prove part (b). Let L^1 have an *inseparable pair* (b, a) . In view of Example 1, Theorem 1 and L^1 being a *maximal Maskin monotone possibility domain* it follows for all $x \in A - \{b\}$ that $L_x(A) \subseteq L^1$ and $L_b^1 = \{R \in L(A) : ba... = R\}$. So, for $(x, y) \neq (a, b)$, with $x \neq y$, there are $R \in L^1$ such that $y...x = R$. Now, $(x, y) \in (Y \cup Z) \times Y$ would, by $Maskin(((Y \cup Z) \times Y) \cup I_A, L^1)$,

imply that $\{x\} \times A \subseteq ((Y \cup Z) \times Y)$ and therewith the contradiction $A \subseteq Y$. In consequence, $((Y \cup Z) \times Y) = \{(a, b)\}$. Hence, $\{a\} = Z$ and $\{b\} = Y$. This completes the proof of part (b).

In order to prove (c) let X_1 and X_2 be a partition of $Y \cup Z$ with $\#X_1 \geq 2$ and $X_1 \times X_2 \subseteq P$. It is sufficient to prove that this contradicts the maximality of L^N . Then $I_A \cup [X_1 \times (Y \cap X_1)] \subsetneq A \times A$ and $I_A \cup [X_1 \times (Y \cap X_1)]$ is transitive. By the definition of P it follows that $X_1 \cap Y \neq \emptyset$. Consider $y \in X_1 \cap Y$ and $x \in X_2$. Then $(x, y) \in (Y \cup Z) \times Y$. Let $R \in L_y^1$ and let $R' \in L(A)$ be such that $R|_{A-\{x\}} = R'|_{A-\{x\}}$ and $y \dots x = R'$. Then $R' \in L^1$ and $\text{Maskin}(((Y \cup Z) \times Y) \cup I_A, L^1)$ would imply $\{x\} \times A \subseteq ((Y \cup Z) \times Y)$ which then contradicts $Y \cup Z \neq A$. So, $R' \notin L^1$. But obviously $\text{Maskin}((X_1 \times (Y \cap X_1)) \cup I_A, L^1 \cup \{R'\})$ which contradicts the maximality of L^N . Next we prove (d).

First we prove $L^1 \subseteq V$. Let $R \in L^1$. Then (1) follows because of the definition of P . In order to show that also (2) is satisfied let $z \dots = R$. Let $T = (\text{non-up}(z, P) - \{z\}) \cap (Y \cup Z)$. Take $x = \text{best}(R|_T)$. So $x \in T$ and for some $y \in Y$ there are R' in L^1 with $y \dots z \dots x \dots = R'$. Because of $z \in Z$, it follows that $(x, z) \notin ((Y \cup Z) \times Y) \cup I_A$. Therefore $\text{Maskin}(((Y \cup Z) \times Y) \cup I_A, L^1)$ implies $\text{low}(x, R') \not\subseteq \text{low}(x, R)$. So, there are $t \in \text{low}(x, R')$ and $x \in \text{low}(t, R)$. Because of the definition of x and the fact that $x \in \text{low}(t, R)$, it follows that $t \notin T$. But then, because $t \in \text{low}(x, R')$, it follows that $t \in \text{non-up}(z, P) - \{z\}$. Now, because $t \notin T$, it implies that $t \notin (Y \cup Z)$. So, $t \in A - (Y \cup Z)$ and $T \subseteq \text{low}(t, R)$ which in turn yields (2).

Next we prove that $V \subseteq L^1$. Because of $L^1 \subseteq V$ and the maximality of L^1 , it is sufficient to prove that V is a *Maskin monotone possibility domain*. So, by Corollary 1 it is sufficient to prove that $\text{Maskin}(((Y \cup Z) \times Y) \cup I_A, V)$, which follows by the previous claim. This completes the proof of part (d) and herewith the only if part.

(If-part) Let Y, Z, P and V as formulated in the if part of the theorem. Let $L^1 = V$. It is sufficient to prove that L^N is a *Maskin monotone possibility domain* and that for all *Maskin possibility domains* \hat{L}^N with $V \subseteq \hat{L}^1$ and $\hat{L}^i = L(A)$, we have that $V = \hat{L}^1$. Note that $P = \{(a, b) \in A \times A : a \neq b \text{ and for all } y \in Y \text{ and all } R \in L_y^1 \dots a \dots b \dots = R\}$. By Claim 1 we have $\text{Maskin}(((Y \cup Z) \times Y) \cup I_A, L^1)$. Setting D of Lemma 1 equal to $((Y \cup Z) \times Y) \cup I_A$, it follows that L^N is a *Maskin monotone possibility domain*. Therefore it is sufficient to prove that there are no $\emptyset \neq D \subsetneq A \times A$ and $V \subsetneq W \subseteq L(A)$ with $\text{Maskin}(D, W)$. To the contrary, let there be such D and W . Without loss of generality we may assume that W is a *maximal Maskin monotone possibility*

domain. By the only-if-part, we may assume the existence of disjoint $Y', Z' \subseteq A$ with $Y' \neq \emptyset$, $Y' \cup Z' \neq A$, an asymmetric and transitive relation P' on A with $[(Y' \cup Z') \times (A - (Y' \cup Z'))] \subseteq P' \subseteq [(Y' \cup Z') \times (A - Y')]$, $\#(Y' \cup Z') \geq 2$, if L^1 has a separable pair (b, a) then $Y' = \{b\}$ and $Z' = \{a\}$, there is no partition X'_1, X'_2 of $Y' \cup Z'$ with $\#X'_1 \geq 2$, $X'_1 \times X'_2 \subseteq P'$ and $W = \{R \in L(A) :$

1. if $y... = R$ for some $y \in Y'$, then $P' \subseteq R$;
2. if $z... = R$ for some $z \in Z'$, then there are $t' \in A - (Y' \cup Z')$ such that for all $x' \in Y' \cup Z'$, with $x' \neq z$ and $(x', z) \notin P'$, $x' \in \text{low}(t', R)\}$.

Now by Claim 1 it follows that $\text{Maskin}(((Y' \cup Z') \times Y') \cup I_A, W)$. Because of $V \subseteq W$, this implies $\text{Maskin}(((Y' \cup Z') \times Y') \cup I_A, V)$. Next we prove that $(((Y' \cup Z') \times Y') \cup I_A) = (((Y \cup Z) \times Y) \cup I_A)$.

First, we prove that $(((Y' \cup Z') \times Y') \cup I_A) \subseteq (((Y \cup Z) \times Y) \cup I_A)$. It is sufficient to prove that $Y' \cup Z' \subseteq Y \cup Z$ and that $Y' \subseteq Y$. In order to prove the former let $x \in A - (Y \cup Z)$. It is sufficient to show that $x \notin Y' \cup Z'$. Clearly by the definition of V it follows that $L_x(A) = V_x$. Because of $V \subseteq W$, it follows that $V_x \subseteq W_x$. Hence, $L_x(A) = W_x$. Suppose $x \in Y'$. Then there are $z \in (Y' \cup Z') - \{x\}$ and $R \in L_x(A) = W_x$ such that $x...z = R$. Now by the assumptions on Y', Z', P' and W we have that $(Y' \cup Z') \times (A - (Y' \cup Z')) \subseteq P \subseteq R$. As $x...z = R$, $x \notin Y'$ and $z \in Y' \cup Z'$ this can only hold if $A = Y' \cup Z'$ which contradicts our assumptions on Y', Z', P' and W . Suppose $x \in Z'$. Then there are $R \in L_x(A) = W_x$ such that $(Y' \cup Z') \times (A - (Y' \cup Z')) \subseteq R$. Now as there are $t \in A - (Y' \cup Z')$ such that $((\text{non-up}(z, P) - \{z\}) \cap (Y' \cup Z')) \subseteq \text{low}(t, R)$ and there are $y \in Y'$ which clearly are in $((\text{non-up}(x, P) - \{x\}) \cap (Y' \cup Z'))$ we have a contradiction. This proves $Y' \cup Z' \subseteq Y \cup Z$.

In order to prove $Y' \subseteq Y$ let $y \in Y'$. To the contrary assume $y \notin Y$. Then by the inclusion $Y' \cup Z' \subseteq Y \cup Z$ we may conclude that $y \in Z$ and that there are $t \in (A - (Y \cup Z)) \subseteq (A - (Y' \cup Z'))$. Furthermore, there are $R \in V_y$ with $yt... = R$. As $V \subseteq W$, it follows that $R \in W_y$ where $y \in Y'$. But $yt... = R$ with $t \in A - (Y' \cup Z')$ contradicts $[(Y' \cup Z') \times (A - (Y' \cup Z'))] \subseteq P \subseteq R$. Hence, $Y' \subseteq Y$.

Next we prove $(((Y \cup Z) \times Y) \cup I_A) \subseteq (((Y' \cup Z') \times Y') \cup I_A)$. By the proof of the previous inclusion, we may assume that both $(Y' \cup Z') \subseteq (Y \cup Z)$ and $Y' \subseteq Y$.

First we show that $Y = Y'$. To the contrary let $b \in Y - Y'$. Now for all $c \in Y' \subseteq Y$, there are preferences R and R' in V such that $cb... = R$ and $bc... = R'$. If $b \notin (Y' \cup Z')$, then $R \in V \subseteq W$ violates condition 1 of W . So, $b \in (Y' \cup Z')$ which means that $b \in Z'$. Because of $R \in W$, it follows that $(x, b) \notin P'$ for all $x \in A - \{c, b\}$. Now, because $R' \in W$, in view of condition 2 of W it follows that $(c, b) \in P'$. Note that for all $x \in Y - \{b, c\}$, there are preferences $R'' = xbc...$ in $V \subseteq W$. So, because of $(c, b) \in P'$, it follows that $Y' = \{c\}$. Since $\{c, b\} \subseteq Y$, it follows that (c, b) is not an *inseparable pair* in V . Therefore, there are $x \in A - \{c, b\}$ and preferences $R''' = cx...b...$ in $V \subseteq W$. Then $Maskin(((Y' \cup Z') \times Y') \cup I_A, W)$ implies $x \in (Y' \cup Z')$. As $Y' = \{c\}$ this yields $x \in Z'$. Because of $R''' \in V$ and $c \in Y$, it follows that $(y, x) \notin P$ for all $y \in A - \{c, x\}$. But then, there are preferences $R'''' = bcx...$ in $V \subseteq W$. Note that although $b, x \in Z'$ and $(x, b) \notin P'$, there is no $t' \in A - (Y' \cup Z')$ such that $b...t'...x... = R''''$ contradicting condition 2 of W . In consequence, $Y = Y'$.

In the following, we prove that $Y \cup Z = Y' \cup Z'$. To the contrary let $Y \cup Z \neq Y' \cup Z'$. We will prove that $[(Y' \cup Z') \times ((Y \cup Z) - (Y' \cup Z'))] \subseteq P$ which because of $\#(Y' \cup Z') \geq 2$ clearly contradicts the assumptions on P . So let $x' \in (Y' \cup Z')$ and let $z \in ((Y \cup Z) - (Y' \cup Z'))$. Because of $Y = Y'$ it follows that $z \in Z - Z'$. Suppose to the contrary that $(x', z) \notin P$. Then there are $y' \in Y$ and $R \in V$ with $y'...z...x'... = R$. As $Y = Y'$ both x' and y' are in $(Y' \cup Z')$. $Maskin(((Y' \cup Z') \times Y') \cup I_A, W)$ implies the contradiction $z \in (Y' \cup Z')$. So, $(x', z) \in P$ and herewith $[(Y' \cup Z') \times ((Y \cup Z) - (Y' \cup Z'))] \subseteq P$.

So, $Y = Y'$ and $Z = Z'$. Next we show that $P = P'$ which then by the definition of V and W yields the desired result that $V = W$. First we show that $P \subseteq P'$. To the contrary, suppose that $(a, b) \in P$ and $(a, b) \notin P'$. Because $P \cup P' \subseteq [(Y \cup Z) \times (A - Y)]$ and $((Y \cup Z) \times (A - (Y \cup Z))) \subseteq P \cap P'$, it follows that $a \in Y \cup Z$ and $b \in Z$. Now because of the definition of V there are $R \in V$ with $bat... = R$ for some $t \in (A - (Y \cup Z))$. Since $V \subseteq W$, this clearly yields a contradiction with condition 2 of W . So, $P \subseteq P'$.

Next we prove $P' \subseteq P$. Suppose $(a, b) \notin P$. We prove that $(a, b) \notin P'$. Because $(a, b) \notin P$ it follows by the definition of V that there are $R \in V$ such that both $y... = R$ for some $y \in Y$ and $...b...a... = R$. As $R \in V \subseteq W$. This shows that $(a, b) \notin P'$. ■

7 Conclusion

By restricting the domain of only one agent, we showed that it is possible to escape the negative conclusions of the Gibbard-Satterthwaite and the Muller-Satterthwaite theorems. Obviously, the social choice functions we construct that are *strategy-proof*, *Pareto-efficient* and *non-dictatorial/Maskin monotonic*, *Pareto-efficient* and *non-dictatorial*, indeed have a dictatorship flavor because of their hierarchical structures.¹¹

A question of interest would be to study how these social functions evolve as we restrict the preferences of more than one agent. We leave this question open for future research.

We close the discussion with two examples. The first one shows that there are domains which admit *non-dictatorial*, *Pareto-efficient* and *strongly positively associated* choice rules but are not *non-dictatorial*, *Pareto-efficient* and *strategy-proof* choice rules.

Example 2 Let $B \subsetneq A$, with $\#B \geq 3$, and let $t \in A - B$. Take $L^i = L(A)$ for $i \geq 2$ and $L^1 = \{R \in L(A) : \text{if for some } b \in B R \in L_b(A), \text{ then there are } c \in B - \{b\} \text{ such that } bct... = R\}$. Define the choice rule K from L^N to A for a profile p as follows: $K(p) = a$ if $x... = p(1)$, $x = a$ and $x \notin B$ else $xyt... = p(1)$ with $\{a, b\} = \{x, y\} \subsetneq B$ and $...a...b... = p(2)$.

Clearly K is *non-dictatorial*, *Pareto-efficient* but not *strategy-proof*. Furthermore, K is *strongly positively associated*. To show this consider $\{i\}$ -deviations p and q in L^N such that $p(i)|_{A-\{a\}} = q(i)|_{A-\{a\}}$ and $\text{low}(a, p(i)) \subseteq \text{low}(a, q(i))$. It is sufficient to show that $K(q) \in \{a, K(p)\}$. If $i > 2$, then $K(p) = K(q)$. Let $i = 2$. If $K(p) \notin B$, then $K(p) = K(q)$. Let $K(p) \in B$. Then $\{K(p), b\} = \{x, y\} \subsetneq B$ for some b, x, y in A and $xyt... = p(1)$ and $...K(p)...b... = p(2)$. If $a \neq b$, then $...K(p)...b... = q(1)$ and consequently $K(q) = K(p)$. If $b = a$, then obviously $K(q) \in \{K(p), a\}$. Let $i = 1$. Let $K(p) \notin B$. If not $a... = q(1)$, then $K(q) = K(p)$. If $a... = q(1)$, then $aK(p)... = q(1)$ and as $K(p) \notin B$ it follows that $a \notin B$. So, in that case $K(q) = a$. Let $K(p) \in B$. Then for some $x, y, b \in B$ we have $\{K(p), b\} = \{x, y\}$, $xyt... = p(1)$ and $...K(p)...b... = p(2)$. If $K(p) = a$, then $p(1) = q(1)$ or $q(1) = yxt...$ and $K(p) = K(q)$. If $a \neq K(p)$ and not $a... = q(1)$, then $K(p) = K(q)$. If $a \neq K(p)$ and $a... = q(1)$, then $aK(p)bt... = q(1)$. So, in that case $a \notin B$ and $K(q) = a$.

¹¹Indeed, at many preference profiles, the agent with a restricted set of preferences gets his top alternative.

Our last example shows that if we restrict the sets of preferences of exactly two agents, then neither the condition of *inseparable pair* nor the condition of *inseparable set* is a necessary condition for the maximality of a *strategy-proof* possibility domain.

Example 3 Let $A = \{x, y, z, t\}$ and $N = \{1, 2\}$. Take $L^1 = \{R \in L(A) : R = x... \text{ or } R = y...x...z... \text{ or } R = t...x...z... \text{ or } R = z...\}$ and $L^2 = \{R \in L(A) : R = x...z...t... \text{ or } R = y...t...x... \text{ or } R = z...x \text{ or } R = t...y...x...\}$. Now define the choice rule K for an arbitrary profile $p \in L^N$ as follows $K(p) = a$ if $p(1) = a...$ and $p(2) = b...$ and $(a, b) \notin \{(y, x), (t, x)\}$ else take $K(p) = x$. So at K agent 2 is decisive on the pairs (x, y) and (x, z) , whereas agent 1 is decisive on the remaining pairs. So K is not dictatorial. As $K(p) \in \{best(p(1)), best(p(2))\}$ we have that K is Pareto-efficient. To see that K is strategy-proof consider profiles p with $p(2) = x...$. Then the outcome for any $\{1\}$ -deviation of p is either x or z depending on whether agent 1 prefers x to z or z to x respectively. A similar reasoning holds for $\{2\}$ -deviation of profile p with $p(1) = y...$ or $p(1) = t...$. It is straight forward to see that for both agents i , the set L^i does not have an inseparable pair or set. Therefore, we may conclude that these inseparabilities are not necessary conditions for maximal strategy-proof possibility domains in case the set of preferences of more than one agent is restricted. Note further that K is tops-only but that the sets L^1 and L^2 are neither convex nor connected.

8 References

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